Diskrete Mathematik

Exercise 5

5.1 Family relations (∗ ∗)
Consider the set of all people (both living and already dead) and different family relations on this set: id (identity), if (is the father of), im (is the mother of), ip (is a parent of) and ic (is a child of).

a) Express each of the following relations, using the relations above.
   i) \( x \ iGGf \ y \iff x \) is a great grandfather of \( y \)
   ii) \( x \ iHS \ y \iff x \) is a half sibling of \( y \) (i.e., \( x \) and \( y \) have exactly one common parent)
   iii) \( x \ iCo \ y \iff x \) is a cousin of \( y \) (i.e., \( x \) and \( y \) are not siblings but have a common grandparent)

b) What is the relationship between the relations \( ic \circ ic \circ ip \circ ip \) and \( ic \circ ip \circ ic \circ ip \)? Are they the same relation or is one of them a subset of the other?

5.2 Operations on relations (∗ ∗)
Let us consider the relations \(<, |\) and \(\equiv_2\) on the set of natural numbers \(\mathbb{N} \cup \{0\}\). For each of the following relations on \(\mathbb{N} \cup \{0\}\), decide whether it is reflexive, symmetric or transitive. Justify your answers.
   a) \(< \circ |\)
   b) \(| \cup \equiv_2\)
   c) \(| \cup |^{-1}\)

5.3 Properties of relations (7 Points)

a) (∗ ∗) For the relation \(\rho = \{(1, 4), (2, 1), (2, 3), (4, 2)\}\) on the set \(\{1, 2, 3, 4\}\), determine the relations \(\rho^3\) and \(\rho^*\). Describe \(\rho^3\) using the set representation, while \(\rho^*\) using matrix representation. (2 Points)

b) (∗ ∗) Prove or disprove the following statement: for any set \(A\), if a relation \(\sigma\) on \(A\) is not reflexive, then the relation \(\sigma^2\) is also not reflexive. (2 Points)

c) (∗ ∗ ∗) Prove or disprove the following statement: for any set \(A\), if relations \(\sigma\) and \(\rho\) on \(A\) are antisymmetric, then so is the relation \(\sigma \cap \rho\). (3 Points)
5.4 A false proof (⋆ ⋆)
Consider a non-empty set $A$ and a symmetric and transitive relation $\rho \neq \emptyset$ on $A$.

a) The following proof shows that $\rho$ is always reflexive. Find the mistake in this proof.

Proof: We show that $\rho$ is reflexive, that is that for any $x$, we have $x \rho x$. Let $x \in A$. Further, let $y \in A$ be such that $x \rho y$. Since $\rho$ is symmetric, it follows that $y \rho x$. Now we have $x \rho y$ and $y \rho x$. Hence, by the transitivity of $\rho$, it follows that $x \rho x$.

b) Show that the above statement is indeed false, that is, prove that $\rho$ is not always reflexive.

5.5 Equivalence Relations
For an $x \in \mathbb{R}$, let us define the following relation $\sim$ on $\mathbb{R}^2$:

$$(a, b) \sim (c, d) \iff ((a - x)^2 + b^2)((c + x)^2 + b^2) - ((a - x)^2 + b^2)((c + x)^2 + d^2) = 0$$

a) (⋆ ⋆) Show that $\sim$ is an equivalence relation.

b) (⋆ ⋆) Describe geometrically the equivalence classes $[(a, b)]$.

Due on 23. October 2017.
Exercise 5.3. will be corrected.