Exercise 3

Part 1: Predicate logic

3.1 Quantifiers and predicates (10 Points)

In this exercise the universe is fixed to the set $\mathbb{Z}$ of integers.

a) For each of the following statements, give a formula where the only predicates are \textit{less}, \textit{equals} and \textit{prime} (instead of \textit{less}(n,m) and \textit{equals}(n,m) you can write $n < m$ and $n = m$ accordingly). You can also use the symbols $+$ and $\cdot$ to denote addition and multiplication.

i) (⋆) If the product of two integers is positive, then at least one of these numbers is positive. (2 Points)

ii) (⋆) For every natural number, one can find a strictly greater natural number that is divisible by 3. (2 Points)

iii) (⋆ ⋆) Every even integer greater than 2 is the sum of two primes. (2 Points)

Bonus question: Which of the above statements are true?

b) Consider the following predicates $P(x)$ and $Q(x,y)$:

$$P(x) = \begin{cases} 1, & x > 0 \\ 0, & \text{otherwise} \end{cases} \quad Q(x,y) = \begin{cases} 1, & xy = 1 \\ 0, & \text{otherwise} \end{cases}$$

In this context, describe the following formulas as mathematical statements (that is, write them using words). Also, for each of the formulas decide whether it is true or false.

i) (⋆) $\forall x \exists y Q(x,y)$ (2 Points)

ii) (⋆) $\exists x (\forall y \neg Q(x,y) \land \exists y P(y))$ (2 Points)

3.2 Transitivity of quantifiers

Prove that:

a) (⋆) $\exists y \forall x P(x,y) \models \forall x \exists y P(x,y)$. 

Disprove that:

b) (⋆) $\forall x \exists y P(x,y) \models \exists y \forall x P(x,y)$. 

3.3 Winning strategy (∗ ∗)

Alice and Bob play a game in which the stake is a chocolate bear. Rules of the game are the following: Alice chooses two integers $a_1, a_2$ and Bob chooses two integers $b_1, b_2$. Alice wins whenever $a_1 + (a_2 + b_1)b_2 + 1 = 1$ and Bob wins otherwise.

a) First, consider the case when Alice and Bob announce all their numbers at the same time. Give a formula that describes the statement “Alice has a winning strategy.” Is this statement true?

b) In the second case, Alice and Bob declare their numbers one by one. That is, first Alice announces $a_1$, then Bob announces $b_1$, afterwards Alice announces $a_2$, and at the end Bob replies with $b_2$. Once again, give a formula that describes the statement “Alice has a winning strategy.”. Is this statement true in this case?

Part 2: Proof techniques

3.4 Direct Proof of an Implication (2.4.3)

Prove directly:

a) (∗) The product of two even natural numbers is even.

3.5 Indirect Proof of an Implication (2.4.4)

Prove indirectly that for all natural numbers $n > 0$, we have:

a) (∗ ∗) If $42^n - 1$ is a prime, then $n$ odd.

b) (∗ ∗) If $n^2$ is odd, then $n$ is also odd.

3.6 Case Distinction (2.4.7)

Prove by case distinction that:

a) (∗ ∗) For all integers $n$, we have that $5n^2 + 3n + 42$ is even.

b) (∗ ∗ ∗) If $p$ and $p^2 + 2$ are primes, then $p^3 + 2$ is also a prime.


Exercise 3.1. will be corrected.