14.1 Statements about formulas

a) This expression is a syntactically correct formula.

b) This is a statement about the formulas $\forall x P(x)$ and $P(x)$. It is true. Let $A$ be an interpretation suitable for both $\forall x P(x)$ and $P(x)$, which is a model for $\forall x P(x)$. Since $A(\forall x P(x)) = 1$, it follows that for all $u \in U^A$, $A[x \mapsto u](P(x)) = 1$. Hence, no matter which $x^A \in U^A$ is assigned to $x$ by $A$, $A(P(x)) = 1$. Therefore, $A$ is also a model for $P(x)$.

c) This is not a valid expression, since $\equiv$ can only be used between formulas and $P(x)$ is a statement, not a formula.

d) This is a statement about formulas and it is false. As a counterexample, consider the structure: $U^A = \{0, 1\}$, $P^A(x) = 1 \iff x = 1$, $f^A(x) \equiv 1$, $a^A = 0$. Then we have $A(P(x)) = 1$ and $A(P(f(a))) = 1$, but $A(P(a)) = 0$.

14.2 Relation between validity of a formula and statement about formulas

a) $F := (G_1 \land \cdots \land G_k) \rightarrow H$.

$F$ is a tautology if and only if $A((G_1 \land \cdots \land G_k) \rightarrow H) = 1$ for all suitable interpretations $A$. This means that if $A(G_1 \land \cdots \land G_k) = 1$, then we must have $A(H) = 1$ as well. That is, for any interpretation $A$, such that $A(G_1) = \cdots = A(G_k) = 1$, we have $A(H) = 1$. This means exactly that $\{G_1, \ldots, G_k\} \models H$.

Note: This is only an example, any formula equivalent to $F$ would also be a good solution.

b) $F := G \leftrightarrow H$.

$F$ is a tautology if and only if $A(G \leftrightarrow H) = 1$ for all suitable interpretations $A$. This means that if $A(G) = 1$, then $A(H) = 1$ as well, and if $A(H) = 1$, then $A(G) = 1$ as well. Hence, by the definition of $\models$, we have $G \models H$ and $H \models G$. Therefore, $G \equiv H$.

Note: Again, this is only an example.

14.3 Calculi

a) The following rules are correct: $R_1$, $R_2$, $R_4$, and $R_6$.

To show this, for each rule $R$ we consider the statement $M \models H$ for a set $M$ and a formula $H$. If this statement is true for any $M$ and $H$ such that $M \vdash_R H$, then the rule is correct. We show $M \models H$ by drawing a function table and checking that the
truth value of $H$ is 1 whenever the truth values of all formulas in $M$ are 1. A rule is incorrect if the statement $M \models H$ is false. We show this by giving a counterexample (the counterexamples are the rows in the corresponding function tables, printed in bold).

<table>
<thead>
<tr>
<th>$F$</th>
<th>$G$</th>
<th>$F \lor G$</th>
<th>$F$</th>
<th>$G$</th>
<th>$F \land G$</th>
<th>$F$</th>
<th>$G$</th>
<th>$\neg(F \land G)$</th>
<th>$\neg F \land \neg G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$F$</th>
<th>$G$</th>
<th>$F \rightarrow G$</th>
<th>$F$</th>
<th>$G$</th>
<th>$F \rightarrow \neg G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

b) We have $K = \{R_1, R_2, R_4, R_6\}$. The derivation is the following:

\[
\{B \land A\} \vdash_{R_2} B \\
\{B\} \vdash_{R_4} B \lor C \\
\{B \lor C; (B \lor C) \rightarrow D\} \vdash_{R_4} D \\
\{A \land B\} \vdash_{R_2} A \\
\{D, A\} \vdash_{R_6} D \land A \\
\{D \land A; (D \land A) \rightarrow C\} \vdash_{R_4} C \\
\{A \land B, C\} \vdash_{R_6} A \land B \land C \\
\{A \land B \land C, D\} \vdash_{R_6} A \land B \land C \land D
\]

c) The calculus $K = \{R_2, R_4\}$ is not complete. As a counterexample, consider the set $M_0 = \{A \land B\}$ and the formula $H := B \land A$. We have $A \land B \models B \land A$. However, $H$ cannot be derived from $M_0$. Indeed, to $M_0$ one can only apply $R_2$ with $F := A$ and $G := B$, obtaining the set $M_1 = \{A \land B, A\}$. But no new formulas can be derived from $M_1$.

d) For example, the following calculus $K := \{R\}$ with $\emptyset \models_R F$ is complete but not sound.

In the calculus $K$, one can derive exactly all formulas. Hence, it is clearly complete. It is also clearly not sound, since for example, the formula $A \land B$ can be derived and it is not a tautology.

14.4 Resolution in propositional logic

a) i) The clauses are: $\{A, B\}, \{\neg E\}, \{\neg B, D\}, \{\neg D, E\}, \{\neg A, B\}$. 
Hence, the formula is not satisfiable.

ii) The formula 
\[ G = (\neg B \land \neg C \land D) \lor (\neg B \land \neg D) \lor (C \land D) \lor B \]

is a tautology if and only if 
\[ \neg G \equiv (B \lor C \lor \neg D) \land (B \lor D) \land (\neg C \lor \neg D) \land (\neg B) \]

is not satisfiable. We show this, using the resolution calculus:

iii) Let \( \mathcal{K}(M) = \{\{\neg A, C\}, \{A, \neg B\}, \{A, B\}\} \) be the set of clauses, corresponding to the set \( M \). The set of clauses corresponding to \( \neg H \) is \( \mathcal{K}(\neg H) = \{\neg A, \neg C\} \). We show that \( \mathcal{K}(M) \cup \mathcal{K}(\neg H) \) is unsatisfiable.

b) There is only a finite number of atomic formulas in \( \mathcal{K} \). Let \( k \) denote their number. Since in a clause an atomic formula can either: appear plain, appear negated, appear in both forms or not appear at all, the number of possible clauses that can be derived from \( \mathcal{K} \) is \( 4^k \). Now for all \( i \geq 0 \), we have \( \mathcal{K}_i \subseteq \mathcal{K}_{i+1} \). It follows that \( |\mathcal{K}_i| \leq |\mathcal{K}_{i+1}| \), which, together with the fact that \( |\mathcal{K}_i| \leq 4^k \), implies that for some \( n \geq 0 \), we have \( |\mathcal{K}_n| = |\mathcal{K}_{n+1}| = \ldots \). It follows that no new clauses can be added, that is, \( \mathcal{K}_n = \mathcal{K}_{n+1} = \ldots \).

c) For \( i \in \mathbb{N} \), let 
\[ \mathcal{K}_i := \mathcal{K} \cup \bigcup_{j=1}^{i} \{\{A_0, \neg A_{j+1}\}\}. \]

Graphically, the constructed sequence of derivations looks as follows:
More formally, we clearly have $K_0 = \mathcal{K}$ and $K_i \neq K_{i-1}$ for all $i > 0$. What is left to show is that for all $i > 0$, there exist $K', K'' \in K_{i-1}$ and $K_i$ such that $\{K', K''\} \vdash_{\text{res}} K$ and $K_i = K_{i-1} \cup \{K\}$ (where $K$ is the new clause, $K \notin K_{i-1}$). Indeed, for any $i > 0$, we can take $K' = \{A_0, \neg A_i\} \in K_{i-1}$ and $K'' = \{A_i, \neg A_{i+1}\} \in \mathcal{K} \subseteq K_{i-1}$. Then we have $\{K', K''\} \vdash_{\text{res}} \{A_0, \neg A_{i+1}\}$ (so $K = \{A_0, \neg A_{i+1}\}$) and

$$K_i := \mathcal{K} \cup \bigcup_{j=1}^{i} \{A_0, \neg A_{j+1}\} = \mathcal{K} \cup \bigcup_{j=1}^{i-1} \{A_0, \neg A_{j+1}\} \cup \{A_0, \neg A_{i+1}\} = K_{i-1} \cup \{K\}.$$