11.1 Conditional Probability Distributions

Goal: We repeat the basics on conditional probability distributions and how they are composed to define a random experiment.

In this task, we define a random experiment using a uniform random permutation $P_n$ as defined in the lecture notes. We consider an environment $E$ that issues two queries to $P_n$ and then stops. More specifically, $E$ chooses the first input uniformly and sets the second input to be equal to the first output of $P_n$ and then stops after receiving the reply from $P_n$. The random experiment defined by $E$ and $P_n$ then evolves as follows:

1. $E$ chooses the first input $X_1$ uniformly at random from $\{0, 1\}^n$.
2. $P_n$ obtains the input $X_1$ and responds with the output $Y_1$.
3. $E$ obtains the output $Y_1$ and then sets $X_2 = Y_1$.
4. $P_n$ obtains the input $X_2$ and, based on $X_1$, $Y_1$, and $X_2$, responds with the output $Y_2$.
5. $E$ outputs $\perp$.

a) Describe the behavior of $E$ in terms of conditional probability distributions and compute the resulting distribution $Pr_{X_1 X_2 Y_1 Y_2}$ of the transcript.

b) What is the distribution of the outputs $Y_1$ and $Y_2$ of $P_n$, i.e., what is $Pr_{Y_1 Y_2}$? What is the conditional distribution $Pr_{Y_1 Y_2 | X_1 X_2}$?

11.2 Distinguishing URFs and URPs

Goal: In this exercise, we complete the proof of Lemma 4.19.

a) Prove the statement from Example 4.15, i.e., for the uniform random function $R_{n,n}$ and the uniform random permutation $P_n$, formalize an MBO $A_1, A_2, \ldots$ such that $R_{n,n} \equiv P_n$ and prove the conditional equivalence.

b) Prove Lemma 4.18. In more detail, let $X_1, \ldots, X_q$ be uniformly-distributed independent random variables on some set $\mathcal{X}$ with $|\mathcal{X}| = t$. Denote by $p_{\text{coll}}(q, t)$ the probability that
there exists a collision, i.e., there exist indices $i, j$ with $1 \leq i < j \leq q$ and $X_i = X_j$. Show that

$$p_{\text{coll}}(q, t) \leq \frac{1}{2} q^2 / t.$$ 

**Hint:** What is the probability (for some $i \neq j$) that $X_i = X_j$? How many such pairs $i \neq j$ are there?

### 11.3 Distinguishing Systems Adaptively

**Goal:** To sharpen the view on (non-)adaptive distinguishers, we examine an easy example to see how adaptivity can help.

Consider the following two discrete random systems $S_0$ and $S_1$. System $S_0$ accepts $n$-bit strings as inputs and, upon receiving such an input, ignores its value and returns a uniformly distributed random $n$-bit string. System $S_1$ behaves similarly to $S_0$, but whenever an input is equal to one of the previous outputs, it outputs a special fixed symbol $\perp$.

Assume that you are given either $S_0$ or $S_1$. Your task is to devise a distinguisher $D$ that tries to distinguish $S_0$ and $S_1$ by providing inputs to the system and seeing the corresponding output values.

**a)** Is it possible to distinguish the two systems $S_0$ and $S_1$ if you are only allowed to provide a single input to the system?

**b)** What is the best strategy if you are allowed any number of queries?

**c)** What happens if you have to fix all your inputs in advance (before seeing any output)?

### 11.4 Expansion of PRGs

**Goal:** We analyze a construction to expand pseudo-randomness and use the Constructive Cryptography framework.

Let $g : \{0, 1\}^k \rightarrow \{0, 1\}^{2k}$ be a function (think of a PRG). We want to use this function to generate $4k$ (pseudo-random) bits from a $k$-bit seed and formulate it as a construction (cf. Example 5.1 of the lecture notes).

Let $G$ be the resource that on the first input $s \in \{0, 1\}^k$ returns $g(s)$ (and ignores subsequent inputs). Let further denote $U_n$ the resource that upon the first invocation outputs a uniformly distributed random $n$-bitstring. Finally, let $\alpha[U_k, G]$ be the resource that on the first activation outputs $g(s)$ for a uniformly random $k$-bitstring $s$ (implemented by a converter $\alpha$ that routes the output of $U_k$ as input to $G$).

**a)** Describe the specification that we aim to construct (using Section 5.3.5 from the lecture notes) as a generic relaxation of $\{U_{4k}\}$ that contains all systems $S$ such that the distinction problem $\langle \alpha[U_k, G] \mid U_{2k} \rangle$ reduces to the distinction problem $\langle U_{4k} \mid S \rangle$ (for some reduction $\rho$ with performance-translation $\lambda$).

**b)** Describe the assumed specification $R$ based on the above resources. Then give a converter $\beta$ and show which specification $S$ (of the type defined in a)) is constructed (cf. Definition 5.4 of the lecture notes) by providing the concrete reduction and performance-translation functions.

**Hint:** Think of the following construction: compute $s_1|s_2 := g(s)$ ($s_1, s_2 \in \{0, 1\}^k$) and output $g(s_1)|g(s_2)$. Note that $R$ can also be a singleton set.

**Discussion of solutions:**

Tuesday, 22.5.2018 (Tasks 11.1, 11.2, and 11.3)

28/29.5.2018 (Task 11.4)

The Monday and Tuesday sessions of each week cover the same material.