Nonlocality is Transitive

Sandro Coretti

Computer Science Department, ETH Zurich, CH-8092 Zürich, Switzerland

Esther Hänggi

Centre for Quantum Technologies, National University of Singapore, Singapore

Stefan Wolf

Computer Science Department, ETH Zurich, CH-8092 Zürich, Switzerland
(Received 28 February 2011; published 31 August 2011)

We show a transitivity property of nonlocal correlations: There exist tripartite nonsignaling correlations of which the bipartite marginals between A and B as well as B and C are nonlocal and any tripartite nonsignaling system between A, B, and C consistent with them must be such that the bipartite marginal between A and C is also nonlocal. This property represents a step towards ruling out certain alternative models for the explanation of quantum correlations such as hidden communication at finite speed. Whereas it is not possible to rule out this model experimentally, it is the goal of our approach to demonstrate this explanation to be logically inconsistent: either the communication cannot remain hidden, or its speed has to be infinite. The existence of a three-party system that is pairwise nonlocal is of independent interest in the light of the monogamy property of nonlocality.

Introduction.—In a classical world, correlations between distant observers are either due to preshared information or communication [1]. Shared quantum information, i.e., measurements on two (or more) distant parts of an entangled quantum state, however, can lead to correlations which are, on one hand, stronger than what can be achieved by shared information, but, on the other hand, do not allow for communication. These correlations are called nonlocal [2,3], and the formalism of quantum physics predicts that they occur no matter the distance between the observers nor the position in space-time they perform their measurements at.

While quantum theory is well-established and no experiment has contradicted its predictions, the question remains open whether there could be other physical theories describing our world. The fact that correlations predicted by quantum theory are nonlocal and occur even when measurements are performed in a spacelike separated way [4] implies that such a theory cannot be limited to local hidden variables [3] and communication at the speed of light. Among alternative models which could explain the experimental observations, one proposition is to consider a physical theory based on local hidden variables augmented by superluminal hidden communication (in a preferred reference frame) for transmitting the nonlocal correlations. If this superluminal communication were of infinite speed, this model would be consistent with the predictions of quantum theory; however, is it possible that it occurs at some finite speed (possibly much faster than the speed of light)? It has been pointed out that such communication alone is insufficient [5,6]; but what if it is augmented with hidden variables?

If the cause of nonlocal correlations was hidden communication at finite speed, then the correlations could only be observed between two observers as long as the hidden communication can travel from one to the other; in case the observers measure their systems "too simultaneously," their correlations would have to turn local. Experiments can, therefore, give a lower bound on the required speed of such a communication in a specific reference frame [7]; however, they can never exclude that the hidden communication occurred at an even higher, but still finite, speed. Hence, no experiment can rule out any finite speed.

For that reason, it is our goal to rule out this model in principle, by showing that the assumptions that the communication is both hidden and of finite speed lead to a logical contradiction. This can be done by a Gedankenexperiment [5,6], for which we need to find correlations between three parties Alice, Bob, and Charlie that are "transitive" in the following way: Assume Alice and Bob as well as Bob and Charlie are both close enough for the hidden communication to arrive, i.e., the correlations AB and BC are nonlocal, while Alice and Charlie are far apart (see Fig. 1) [8]. Now, if the marginal bipartite correlations AB and BC are such that any consistent nonsignaling correlation must also be nonlocal between Alice and Charlie, then the speed of any hidden communication would necessarily have to be infinite. Therefore, this "transitivity of nonlocality" rules out finite-speed communication as its explanation in principle, independently of any possible experiment.

Note that under the assumption that the correlations are obtained by measurements on a quantum state ρ_{ABC},
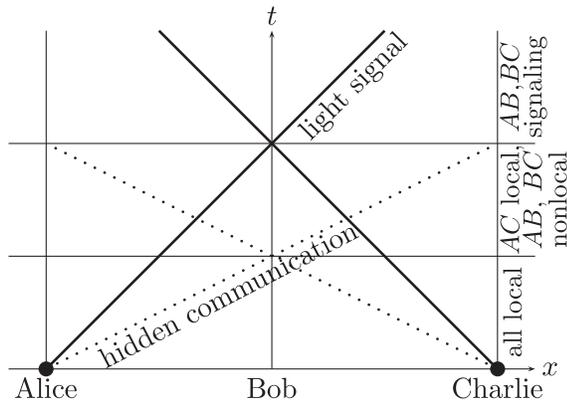
FIG. 1.   The correlations between Alice, Bob, and Charlie, if Alice and Charlie measure their system at $t = 0$. Depending on when Bob performs his measurement, his correlation with them can be either: local (before the hidden communication arrived), nonlocal but nonsignaling (the hidden communication but no signal arrived), signaling (light signal arrived). Our result states that there are nonlocal correlations between Alice and Bob and between Bob and Charlie which, if they could be observed, would exclude the existence of the sector where only AB and BC are nonlocal, whereas AC is local.

nonlocality between Alice and Charlie can be inferred even from correlations between Alice and Bob (ρ_{AB}) and between Bob and Charlie (ρ_{BC}) which are local. The reason is that the bipartite marginals of a state (almost always) determine the full state [9]. However, since it is our goal to compare quantum physics to alternative models, we must reason beyond the quantum formalism, i.e., in terms of input-output systems instead of quantum states.

In summary, it is our goal to find nonlocal correlations AB and BC that imply nonlocality between AC under any nonsignaling composition. We show that such correlations, which we call transitive nonlocal, do indeed exist, even with respect to Bell inequalities with as little as four measurement settings and two outcomes. It is an open question whether the correlations we describe are consistent with quantum theory and whether we could find a tripartite quantum state ρ_{ABC} and measurements which lead to the input-output systems of AB and of BC that imply nonlocality between A and C. This would rule out any explanation of quantum-physically achievable nonlocality based on finite-speed hidden communication.

Preliminaries.—We characterize a tripartite (physical) system by the probabilities of the measurement results given the choice of measurement, i.e., a conditional probability distribution $P_{XYZ|UVW}$, where U and X are Alice's choice of measurement and measurement result, respectively, and similarly, V and Y are associated with Bob, and W and Z with Charlie. The system $P_{XYZ|UVW}$ is called nonsignaling if by interacting with any marginal side no information about the choice of measurement at the other side(s) can be obtained. For example, it is nonsignaling

from Charlie to Alice and Bob if for all x, y, u, v, w, w' it holds that

$$\sum_z P_{XYZ|UVW}(x, y, z, u, v, w) = \sum_z P_{XYZ|UVW}(x, y, z, u, v, w').$$
(1)

We require the system to be nonsignaling between all possible two disjoint subsets of Alice, Bob, and Charlie.

A system is called local deterministic if the output on each side is a deterministic function on the input only on this side, i.e.,

$$P_{XYZ|UVW}(x, y, z, u, v, w) = \delta_{f(u)x}\delta_{g(v)y}\delta_{h(w)z},$$

where $f : \mathcal{U} \mapsto \mathcal{X}$ is a function mapping each input to a fixed output, and similarly for g and h. A system is called local if it is a convex combination of local deterministic systems. Local systems are exactly the ones which can be described by local hidden variables. A system which is nonsignaling, but not local, is called nonlocal.

Note that both the space of nonsignaling as well as the space of local systems form a convex polytope (for any number of inputs and outputs). The polytope of local systems is (in general) strictly contained in the space of nonsignaling systems.

Bell inequalities [3,10] are linear in the probabilities $P_{XYZ|UVW}(x, y, z, u, v, w)$ and fulfilled by any local system. If we write $\vec{p}$ for the vector where the entries are all conditional probabilities $P_{XYZ|UVW}(x, y, z, u, v, w)$, a Bell inequality is of the form $\vec{b}^{\mathsf{T}}\vec{p} \leq c$, where $\vec{b}$ contains the linear coefficients describing the Bell inequality, and c is a scalar. For example, the half-spaces determining the local polytope are Bell inequalities. Conversely, this also implies that any nonlocal system must necessarily violate some Bell inequality. For a system $P_{XYZ|UVW}(x, y, z, u, v, w)$ (not necessarily local) described by $\vec{p}'$, we will say that it reaches a Bell value of c' if $\vec{b}^{\mathsf{T}}\vec{p}' = c'$.

For a given system $P_{XYZ|UVW}(x, y, z, u, v, w)$, the question whether this system is local can be cast as a linear-programming problem, i.e., an optimization problem where the objective function is a linear function of some vector $\vec{x}$, and the constraints are linear equalities or inequalities in $\vec{x}$ (see, e.g., [11] for a good introduction to linear programming). More precisely, by solving

$$\max: \sum_i q_i \qquad s.t.: A \cdot \vec{q} \leq \vec{p} \qquad q_i \geq 0 \quad \text{for all } i,$$

where q_i are the entries of the vector $\vec{q}$ to be optimized over, and the columns of A are all possible local deterministic systems of this number of inputs and outputs. If the optimal value is 1, then the system is local, if it is smaller than 1, it is nonlocal [12] (see also [13]).

Additionally, note that the nonsignaling conditions (1) are linear in the probabilities, more precisely of the form $A_{n-s}\vec{p} = \vec{0}$. The same holds for the conditions defining a probability distribution, i.e., normalization ($A_{\text{norm}}\vec{p} = 1$) and positivity ($p_i \geq 0$ for all i). This implies that the maximum or minimum Bell values reachable by a nonsignaling (or local) system can be calculated by a linear program. For example, the maximum Bell value reachable by a nonsignaling system corresponds to [14]

$$\text{max: } \vec{b}^{\mathsf{T}} \cdot \vec{p} \qquad s.t.: A_{n-s} \cdot \vec{p} = \vec{0}$$

$$A_{\text{norm}} \cdot \vec{p} = 1 \qquad p_i \geq 0 \quad \text{for all } i.$$

On the other hand, we can minimize the Bell value consistent with certain constraints (such as, for example, a fixed marginal) and, therefore, test whether these constraints are sufficient to imply a Bell inequality violation.

Transitivity.—In order to find a system that is transitive nonlocal, we will use the Bell inequalities given in [15,16] as candidates. Note that the best-known Bell inequality—and the only one for the case of two inputs and outputs—the CHSH inequality [10] is monogamous [14], i.e., a nonsignaling system which violates it between Alice and Bob cannot at the same time violate it between Bob and Charlie. Consequently, we need to consider Bell inequalities with a larger number of inputs and/or outputs. In the following, this will be Bell inequalities with binary outcomes but with up to four inputs.

To find a tripartite system which we can then test for transitivity of nonlocality, we proceed as follows. We choose two Bell inequalities, which Alice and Bob as well as Bob and Charlie should violate. We then maximize the sum of the values of these two Bell inequalities twice, subject to the following constraints: (i) Alice, Bob, and Charlie share a tripartite nonsignaling system. (ii) Alice, Bob, and Charlie share a tripartite nonsignaling system of which the marginal of Alice and Charlie is local. If the optimal value obtained in the first optimization is higher than the one obtained in the second optimization, the tripartite system giving rise to this value cannot be local between Alice and Charlie. A complete list of Bell inequalities which have been tested using this approach and imply transitivity of nonlocality can be found in [17].

The above approach tells us when a system between Alice and Charlie must be nonlocal, i.e., must violate some Bell inequality. It does not necessarily imply that there is a specific Bell inequality which must be violated between Alice and Charlie. Nevertheless, we can check whether this is the case by taking the marginal systems of Alice and Bob and Bob and Charlie obtained from the first optimization above, and then minimize the Bell value of any tripartite nonsignaling system consistent with these marginals. An example of such a tripartite nonsignaling system which must even violate a specific Bell inequality is given in Fig. 2 (see also [17]).

$P_{XYZ|UVW}$	111	112	121	122	211	212	221	222
111	2/3	0	0	0	0	0	0	1/3
112	1/3	1/3	0	0	0	0	0	1/3
113	1/3	1/3	0	0	0	0	0	1/3
114	1/3	1/3	0	0	0	0	0	1/3
121	1/3	0	1/3	0	0	0	0	1/3
122	0	1/3	1/3	0	0	0	0	1/3
123	1/3	0	0	1/3	0	0	0	1/3
124	0	1/3	1/3	0	0	0	0	1/3
131	1/3	0	1/3	0	0	0	0	1/3
132	1/3	0	0	1/3	0	0	0	1/3
133	0	1/3	1/3	0	0	0	0	1/3
134	0	1/3	1/3	0	0	0	0	1/3
141	1/3	0	1/3	0	0	0	0	1/3
142	0	1/3	1/3	0	0	0	0	1/3
143	0	1/3	1/3	0	0	0	0	1/3
144	0	1/3	1/3	0	0	0	0	1/3
211	1/3	0	0	0	1/3	0	0	1/3
212	1/3	0	0	0	0	1/3	0	1/3
213	0	1/3	0	0	1/3	0	0	1/3
214	0	1/3	0	0	1/3	0	0	1/3
221	0	0	1/3	0	1/3	0	0	1/3
222	0	0	1/3	0	0	1/3	0	1/3
223	0	0	0	1/3	1/3	0	0	1/3
224	0	0	0	1/3	0	1/3	1/3	0
231	1/3	0	0	0	0	0	1/3	1/3
232	1/3	0	0	0	0	0	0	2/3
233	0	1/3	0	0	0	0	1/3	1/3
234	0	1/3	0	0	0	0	1/3	1/3
241	0	0	1/3	0	1/3	0	0	1/3
242	0	0	1/3	0	0	1/3	0	1/3
243	0	0	0	1/3	0	1/3	1/3	0
244	0	0	0	1/3	0	1/3	1/3	0
311	1/3	0	0	0	1/3	0	0	1/3
312	0	1/3	0	0	1/3	0	0	1/3
313	1/3	0	0	0	0	1/3	0	1/3
314	0	1/3	0	0	1/3	0	0	1/3
321	1/3	0	0	0	0	0	1/3	1/3
322	0	1/3	0	0	0	0	1/3	1/3
323	1/3	0	0	0	0	0	0	2/3
324	0	1/3	0	0	0	0	1/3	1/3
331	0	0	1/3	0	1/3	0	0	1/3
332	0	0	0	1/3	1/3	0	0	1/3
333	0	0	1/3	0	0	1/3	0	1/3
334	0	0	0	1/3	0	1/3	1/3	0
341	0	0	1/3	0	1/3	0	0	1/3
342	0	0	0	1/3	0	1/3	1/3	0
343	0	0	1/3	0	0	1/3	0	1/3
344	0	0	0	1/3	0	1/3	1/3	0
411	1/3	0	0	0	1/3	0	0	1/3
412	0	1/3	0	0	1/3	0	0	1/3
413	0	1/3	0	0	1/3	0	0	1/3
414	68/375	19/125	0	0	19/125	68/375	0	1/3
421	0	0	1/3	0	1/3	0	0	1/3
422	0	0	0	1/3	0	1/3	1/3	0
423	0	0	0	1/3	1/3	0	0	1/3
424	0	0	68/375	19/125	0	1/3	19/125	68/375
431	0	0	1/3	0	1/3	0	0	1/3
432	0	0	0	1/3	1/3	0	0	1/3
433	0	0	0	1/3	0	1/3	1/3	0
434	0	0	68/375	19/125	0	1/3	19/125	68/375
441	0	0	1/3	0	1/3	0	0	1/3
442	0	0	0	1/3	0	1/3	1/3	0
443	0	0	0	1/3	0	1/3	1/3	0
444	0	0	68/375	19/125	0	1/3	19/125	68/375

FIG. 2.   A tripartite nonsignaling system $P_{XYZ|UVW}$. The rows contain the different possible inputs of Alice, Bob, and Charlie and the columns the outputs.

Figure 2 describes a tripartite nonsignaling system. Consider the Bell inequalities I_{4422}^{11} and I_{4422}^{3} from [15] determined by the following coefficients:

$$
\begin{array}{c|cccc}
I_{4422}^{11} & -2 & -1 & -1 & 0 \\
\hline
-2 & 1 & 1 & 1 & 2 \\
-1 & 1 & 0 & 2 & -1 \\
-1 & 1 & 2 & -1 & -1 \\
0 & 2 & -1 & -1 & -1
\end{array} \le 0 \,,
$$

$$
\begin{array}{c|cccc}
I_{4422}^{3} & -2 & -1 & -1 & 0 \\
\hline
-1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & -1 \\
0 & 1 & -1 & 1 & -1 \\
0 & 1 & 0 & -1 & 0
\end{array} \le 0 \,.
$$

We used here the same notation as [15] to describe a Bell inequality of a bipartite system with binary outputs and m inputs; i.e., the table gives the coefficients associated with the probabilities of the first output (which we denote here by 1) of a bipartite system $P_{XY|UV}$ in the following way

$$
\begin{array}{c|ccc}
I & P_{Y|V}(1,1) & \cdots & P_{Y|V}(1,m) \\
\hline
P_{X|U}(1,1) & P_{XY|UV}(1,1,1,1) & \cdots & P_{XY|UV}(1,1,1,m) \\
\vdots & \vdots & \ddots & \vdots \\
P_{X|U}(1,m) & P_{XY|UV}(1,1,m,1) & \cdots & P_{XY|UV}(1,1,m,m)
\end{array}
$$

It follows by a straightforward calculation that the system given in Fig. 2 violates I_{4422}^{11} for the bipartite marginals AB and BC with a value of $2/3$ each. Additionally, AC violates I_{4422}^{3} reaching a value of $1/3$. Minimizing this value for any nonsignaling system consistent with the marginals AB and BC, as obtained from Fig. 2, shows that this is at the same time the minimal value which can be reached. The system given in Fig. 2 is, therefore, transitive nonlocal with respect to the Bell inequality I_{4422}^{3}.

Concluding remarks and open questions.— Measurements on entangled quantum systems can lead to correlations which ask for explanations. Possible such explanations are shared information (so-called hidden variables) or some sort of communication (which would need to be faster than the speed of light, as experiments have indicated—so-called hidden communication). It has been shown by Bell and by Gisin and Scarani, respectively, that one of these two resources is insufficient to explain the correlations in general. We provide strong evidence that this even holds for both combined. More specifically, we show that nonlocal correlations can have some sort of transitivity property: There exist pairs of bipartite correlations between AB and between BC—with identical marginal behavior in B—such that any composition thereof to a three-party nonsignaling system ABC must be such that A and C also share nonlocal correlations. This is incompatible with models where nonlocality is transmitted by finite-speed hidden communication—whatever this speed might

be—as well as models where such correlations exist only up to certain distances. The reason is that such models predict situations where AB and BC are nonlocal, but AC is local—for example, if A and C measure simultaneously but B measures later. We believe that the existence of a three-party system displaying pairwise nonlocality (and of a Bell inequality allowing for this) is of independent interest because of the monogamy property of nonlocality.

It is an open question whether the correlations we consider are quantum-physically realizable, and, in particular, whether there exists a tripartite quantum state ρ_{ABC} whose bipartite marginal systems allow for carrying out a similar reasoning.

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