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Diskrete Mathematik

Exercise 14

14.1 Statements about formulas ($\star\star$)

For each of the following expressions, determine whether it is syntactically correct, and, if so, whether it is a formula or a statement about formulas (whenever parentheses are not necessary, they can be omitted – parentheses do not influence correctness). In case you decide that an expression is a statement about formulas, determine whether it is true or false (each time justify your answer).

a) $\forall x\ \exists y\ (P(z) \leftrightarrow Q(f(f(x,z),y)))$

b) $\forall x\ P(x) \models P(x)$

c) $(P(x) \models P(x)) \equiv Q(x)$

d) $\{P(x), P(f(a))\} \models P(a)$

14.2 Relation between validity of a formula and statement about formulas

The goal of this exercise is to find a formula, which is valid if and only if a statement about a set of formulas is true. For example, the statement $G \models \bot$ is true if and only if the formula $F := \neg G$ is valid. Each time justify your answer.

a) ($\star\star$) Let $G_1, \ldots, G_k, H$ be formulas. Find a formula F which is a tautology if and only if $\{G_1, \ldots, G_k\} \models H$ is true.

b) ($\star\star$) Let G, H be formulas. Find a formula F which is a tautology if and only if $G \equiv H$ is true.

14.3 Calculi

a) ($\star$) Decide which of the following rules are correct (justify your answer):

$$\{F\} \vdash_{R_1} F \vee G \qquad \{F \wedge G\} \vdash_{R_2} F \qquad \{\neg(F \wedge G)\} \vdash_{R_3} \neg F \wedge \neg G$$
$$\{F, F \to G\} \vdash_{R_4} G \qquad \{F \to G\} \vdash_{R_5} \neg F \to \neg G \qquad \{F, G\} \vdash_{R_6} F \wedge G$$

b) ($\star\star$) Let K be the calculus, consisting of those of the rules in Subtask a), which are correct. Using K, derive formally the formula $A \wedge B \wedge C \wedge D$ from the following set of formulas:

$$\{(D \wedge A) \to C,\ A \wedge B,\ B \wedge A,\ (B \vee C) \to D\}$$

c) ($\star\star$) Is $K := \{R_2, R_4\}$ complete? Justify your answer.

d) ($\star\star$) Give an example of a calculus, which is complete but not sound.

14.4 Resolution in propositional logic

a) ($\star$) Prove the following statements, using the resolution calculus.

- **i)** $F := (A \lor B) \land (\neg E) \land (\neg B \lor D) \land (\neg D \lor E) \land (\neg A \lor B)$ is not satisfiable.
- **ii)** $G := (\neg B \land \neg C \land D) \lor (\neg B \land \neg D) \lor (C \land D) \lor B$ is a tautology.
- **iii)** $H := A \land C$ is a logical consequence of $M = \{A \to C, B \to A, A \lor B\}$.

b) ($\star\star\star$) In this exercise you will show that from a finite set of finite clauses, after a finite number of applications of derivation rules, no new clauses can be derived. More specifically, let $\mathcal{K}$ be a finite set of finite clauses and let $\mathcal{K}_0, \mathcal{K}_1, \ldots$ be a sequence of applications of derivation rules, such that $\mathcal{K}_0 = \mathcal{K}$ and $\mathcal{K}_i = \mathcal{K}_{i-1} \cup \{K\}$ for all $i > 0$, where $\{K', K''\} \vdash_{\mathsf{res}} K$ for some $K', K'' \in \mathcal{K}_{i-1}$. Show that there exists an n such that for all $m > n$, $\mathcal{K}_m = \mathcal{K}_n$.

c) ($\star\star\star$) Show that the statement from Subtask b) is no longer true for an infinite set $\mathcal{K}$ of finite clauses. More precisely, let $\mathcal{K} = \{\{A_j, \neg A_{j+1}\} \mid j \in \mathbb{N}\}$. Show that there exists an infinite sequence $\mathcal{K}_0, \mathcal{K}_1, \ldots$, such that $\mathcal{K}_0 = \mathcal{K}$ and $\mathcal{K}_i = \mathcal{K}_{i-1} \cup \{K\}$ for all $i > 0$, where $\{K', K''\} \vdash_{\mathsf{res}} K$ for some $K', K'' \in \mathcal{K}_{i-1}$, and for all $i > 0$, $\mathcal{K}_i \neq \mathcal{K}_{i-1}$.

No handing in.