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Diskrete Mathematik

Solution 13

13.1 Predicate logic with identity

a) The following conditions are necessary and sufficient for a structure $\mathcal{A}$ to be a model for **i)** F and **ii)** G:

i) $|U^{\mathcal{A}}| = 1$

ii) $|U^{\mathcal{A}}| \geq 2$

If $|U^{\mathcal{A}}| = 1$, then clearly for all elements x, y of the universe, we have $x = y$. Hence, the condition **i)** is sufficient. This also means that there do not exist two elements x, y of the universe such that $\neg(x = y)$. Hence, the condition **ii)** is necessary.

If $|U^{\mathcal{A}}| \geq 2$, then there exist two different elements x, y of the universe. Hence, the condition **ii)** is sufficient. This also means that not for all two elements x, y of the universe, we have $x = y$. Hence, the condition **i)** is necessary.

b) An example of such formula H is $\exists x \exists y \exists z \big(\neg(x = y) \wedge \neg(y = z) \wedge \neg(x = z)\big)$.

If $|U^{\mathcal{A}}| \geq 3$, then there exist three different elements x, y, z of the universe. These elements satisfy $\neg(x = y) \wedge \neg(y = z) \wedge \neg(x = z)$.

If $|U^{\mathcal{A}}| < 3$, then, by the pigeonhole principle, at least two among three elements chosen from the universe must be equal. Hence, at least one of $\neg(x = y), \neg(y = z)$ and $\neg(x = z)$ must be false and $\mathcal{A}(H) = 0$.

13.2 Prenex form

The only occurrence of a free variable in F is underlined below:

$$\forall z \exists y \, (P(\underline{x}, g(y), z) \vee \neg \forall x \, Q(x)) \wedge \neg \forall z \exists x \, \neg R(f(x, z), z).$$

In the first step, we find a rectified formula, equivalent to F. That is, we resolve the name collisions. Those collisions are: (1) x appears once as a free variable and twice as a bound variable, and (2) z appears three times as a bound variable. By bound substitution, we get

$$F \equiv \forall z \exists y \, (P(x, g(y), z) \vee \neg \forall t \, Q(t)) \wedge \neg \forall u \exists v \, \neg R(f(v, u), u).$$

In the second step, we apply the equivalences of Lemma 6.7.

$$\begin{aligned} F &\equiv \forall z \exists y \, (P(x, g(y), z) \vee \exists t \, \neg Q(t)) \wedge \exists u \forall v \, R(f(v, u), u) && \text{Lemma 6.7, 1) and 2)} \\ &\equiv \forall z \exists y \exists t \exists u \forall v \big((P(x, g(y), z) \vee \neg Q(t)) \wedge R(f(v, u), u) \big) && \text{Lemma 6.7, 7) to 10)} \end{aligned}$$

13.3 Logical consequence (⋆ ⋆)

a) This statement is true. Let $\mathcal{A}$ be any structure suitable for $G := F \vee \exists y\, \neg F$. We show that $\mathcal{A}$ is a model for G. If $\mathcal{A}(\exists y\, \neg F) = 1$, then clearly $\mathcal{A}$ is a model. If $\mathcal{A}(\exists y\, \neg F) = 0$, then, by the definition of semantics, $\mathcal{A}_{[y \to u]}(\neg F) = 0$ for all $u \in U^{\mathcal{A}}$. Hence, no matter what value $\mathcal{A}$ assigns to y (in case y is not free in F, $\mathcal{A}$ may not assign any value to y), we have $\mathcal{A}(\neg F) = 0$ and $\mathcal{A}(F) = 1$. Thus, $\mathcal{A}$ is a model for G also in this case.

b) This statement is false. As a counterexample, consider the formulas $F = P(x)$ and $G = P(y)$, and the structure: $U^{\mathcal{A}} = \mathbb{N}, P^{\mathcal{A}}(x) = 1 \iff x = 42, x^{\mathcal{A}} = 3, y^{\mathcal{A}} = 5$. Then $\mathcal{A}(\exists x\, P(x) \wedge \exists y\, P(y)) = 1$, because there exist x and y in $\mathbb{N}$, which are equal to 42. However, $\mathcal{A}(P(x) \wedge P(y)) = 0$, because $3 \neq 42$ and $5 \neq 42$.

c) This statement is true. Let $\mathcal{A}$ be any structure suitable for $\forall x\, F, \forall y\, G$ and $F \wedge G$, such that $\mathcal{A}(\forall x\, F) = \mathcal{A}(\forall y\, G) = 1$. By the definition of semantics, this means that $\mathcal{A}_{[x \to u]}(F) = 1$ and $\mathcal{A}_{[y \to v]}(G) = 1$ for all $u, v \in U^{\mathcal{A}}$. Hence, no matter what value $\mathcal{A}$ assigns to x and y (in case x is not free in F or y is not free in G, $\mathcal{A}$ may not assign them any value), we have $\mathcal{A}(F) = 1$ and $\mathcal{A}(G) = 1$. Therefore, $\mathcal{A}$ is also a model for $F \wedge G$.

13.4 The barber of Zürich

By Theorem 6.11,

$$F := \neg \exists x \forall y \left(P(y, x) \leftrightarrow \neg P(y, y)\right)$$

is a tautology, that is, each structure $\mathcal{A}$ suitable for F is a model for F. We define the structure $\mathcal{A}$ as follows: the universe $U^{\mathcal{A}}$ is the set of all people in Zürich and $P^{\mathcal{A}}(x, y) = 1$ if and only if the person y shaves the person x. In this interpretation, the formula F denotes the statement "There does not exist a person x (the barber) in Zürich, such that for every person y in Zürich, x shaves y if and only if y does not shave himself".

13.5 The Exercise 13

a) The sentence can be described as follows:

$$F := \exists x \left(P(x) \to \forall y\, P(y)\right)$$

b)

$$\begin{aligned}
F &\equiv \exists x \left(\neg P(x) \vee \forall y\, P(y)\right) && \text{definition of } \to \\
&\equiv \left(\exists x\, \neg P(x)\right) \vee \left(\forall y\, P(y)\right) && \text{Lemma 6.7 10)} \\
&\equiv \neg\left(\forall x\, P(x)\right) \vee \left(\forall y\, P(y)\right) && \text{Lemma 6.2 1)} \\
&\equiv \neg\left(\forall x\, P(x)\right) \vee \left(\forall x\, P(x)\right) && \text{Lemma 6.9} \\
&\equiv \top && \text{Lemma 6.7 2)/11)}
\end{aligned}$$

c) Let U be the set of all people in a pub, and let P be the predicate, which is true if a given person drinks. F can now be interpreted as follows:

> "There is a person in the pub, such that if this person drinks, then everyone drinks."

Let U be the set of all professors at ETH, and let P be the predicate, which is true if a professor understands his or her field. F can be interpreted as follows:

> "There is a professor at ETH, such that if he or she understands their field, then all professors understand their fields."