ETH Zürich, D-INFK
HS 2017, 31. Oktober 2017

Prof. Ueli Maurer
Marta Mularczyk

# Diskrete Mathematik

# Solution 6

### 6.1 Partial order relations

**a)** **i)** $11$ are $12$ incomparable, since $11 \nmid 12$ and $12 \nmid 11$.

**ii)** $4$ and $6$ are incomparable, since $4 \nmid 6$ and $6 \nmid 4$.

**iii)** $5$ and $15$ are comparable, since $5 \mid 15$.

**iv)** $42$ and $42$ are comparable, since $42 \mid 42$.

**b)** An element $(a, b)$ is smaller than $(2, 5)$ whenever $a \mid 2$ or $a = 2$ and $b \mid 5$. Since $1$ is the only natural number other than $2$ that divides $2$, all pairs $(1, n)$ for $n \in \mathbb{N} \setminus \{0\}$ are smaller than $(2, 5)$ and no pair $(a, n)$ for $a \in \mathbb{N} \setminus \{0, 1, 2\}$ and $n \in \mathbb{N} \setminus \{0\}$ is smaller than $(2, 5)$. What remains is to consider pairs $(a, b)$ where $a = 2$. But then only $1$ and $5$ divide $5$, hence, only the pairs $(2, 1)$ and $(2, 5)$ are smaller than $(2, 5)$.
Therefore, the elements $(a, b)$ such that $(a, b) \leq_{\mathsf{lex}} (2, 5)$ are $(2, 1), (2, 5)$ and $(1, n)$ for $n \in \mathbb{N} \setminus \{0\}$.

**c)** $(\{1, 3, 6, 9, 12\}, |)$ is not a lattice, since $9$ and $12$ do not have a common upper bound.

**d)** We prove that $(A, \preceq^{-1})$ is a poset. To this end, we show that $\preceq^{-1}$ is a partial order on $A$.

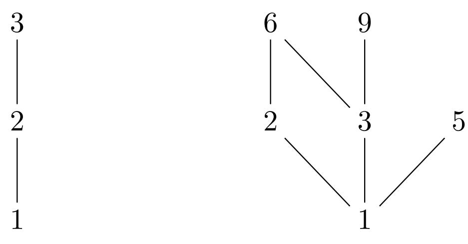
**Reflexivity** For any $a \in A$, by the reflexivity of $\preceq$, we have $a \preceq a$. Therefore, $a \preceq^{-1} a$.

**Antisymmetry:** Let $a, b \in A$ be such that $a \preceq^{-1} b$ and $b \preceq^{-1} a$. This means that $b \preceq a$ and $a \preceq b$ By the antisymmetry of $\preceq$, it follows that $a = b$.

**Transitivity:** Let $a, b, c \in A$ be such that $a \preceq^{-1} b$ and $b \preceq^{-1} c$. This means that $b \preceq a$ and $c \preceq b$. By the transitivity of $\preceq$, we have $c \preceq a$. Hence, $a \preceq^{-1} c$.

### 6.2 Hasse diagram

**a)** The Hasse diagrams of the posets $(\{1, 2, 3\}; \leq)$ and $(\{1, 2, 3, 5, 6, 9\}; |)$ are as follows:




In both cases, $1$ is the least and the only minimal element. In the poset $(\{1, 2, 3\}; \leq)$, the greatest and the only maximal element is $3$. In the poset $(\{1, 2, 3, 5, 6, 9\}; |)$ there is no greatest element. The maximal elements in this poset are $5, 6$ and $9$.

### 6.3 Lexicographic order

For posets $(A; \preceq)$ and $(B; \sqsubseteq)$ the lexicographic order $\leq_{\text{lex}}$ on $A \times B$ is defined by

$$(a_1, b_1) \leq_{\text{lex}} (a_2, b_2) \;:\Longleftrightarrow\; a_1 \prec a_2 \vee (a_1 = a_2 \wedge b_1 \sqsubseteq b_2)$$

We show that $\leq_{\text{lex}}$ fulfills all the properties of a partial order relation.

**Reflexivity** Take any $(a_1, b_1) \in A \times B$. Since $\sqsubseteq$ is reflexive, we have $b_1 \sqsubseteq b_1$. Hence, it is true that $(a_1 = a_1 \wedge b_1 \sqsubseteq b_1)$ and, thus, $(a_1, b_1) \leq_{\text{lex}} (a_1, b_1)$. *(2 Points)*

**Antisymmetry** Take any $(a_1, b_1)$ and $(a_2, b_2)$ in $A \times B$ such that $(a_1, b_1) \leq_{\text{lex}} (a_2, b_2)$ and $(a_2, b_2) \leq_{\text{lex}} (a_1, b_1)$. This means that

$$\underbrace{a_1 \prec a_2}_{(1)} \vee \underbrace{(a_1 = a_2 \wedge b_1 \sqsubseteq b_2)}_{(2)} \quad \text{and} \quad \underbrace{a_2 \prec a_1}_{(3)} \vee \underbrace{(a_2 = a_1 \wedge b_2 \sqsubseteq b_1)}_{(4)}.$$

We have to show that $(a_1, b_1) = (a_2, b_2)$. The proof proceeds by case distinction.

(1) **and** (3)**:** We have $a_1 \preceq a_2 \;\wedge\; a_1 \neq a_2$ and $a_2 \preceq a_1 \;\wedge\; a_2 \neq a_1$. But since $\preceq$ is antisymmetric, it follows that $a_1 = a_2$, which is a contradiction with $a_1 \neq a_2$. Therefore, this case cannot occur.

(1) **and** (4)**:** We have $a_1 \preceq a_2 \wedge a_1 \neq a_2$ and $a_2 = a_1 \wedge b_2 \sqsubseteq b_1$, which is a contradiction. Therefore, this case also cannot occur.

(2) **and** (3)**:** We have $a_1 = a_2 \wedge b_1 \sqsubseteq b_2$ and $a_2 \preceq a_1 \wedge a_2 \neq a_1$, which is a contradiction. Therefore, this case cannot occur as well.

(2) **and** (4)**:** We have $a_1 = a_2 \wedge b_1 \sqsubseteq b_2$ and $a_2 = a_1 \wedge b_2 \sqsubseteq b_1$. Since $\sqsubseteq$ is antisymmetric, it follows that $b_1 = b_2$. But we also have $a_1 = a_2$ and, thus, $(a_1, b_1) = (a_2, b_2)$.

*(3 Points)*

**Transitivity:** Take any $(a_1, b_1), (a_2, b_2), (a_3, b_3)$ in $A \times B$ such that $(a_1, b_1) \leq_{\text{lex}} (a_2, b_2)$ and $(a_2, b_2) \leq_{\text{lex}} (a_3, b_3)$. This means that

$$\underbrace{a_1 \prec a_2}_{(1)} \vee \underbrace{(a_1 = a_2 \wedge b_1 \sqsubseteq b_2)}_{(2)} \quad \text{und} \quad \underbrace{a_2 \prec a_3}_{(3)} \vee \underbrace{(a_2 = a_3 \wedge b_2 \sqsubseteq b_3)}_{(4)}.$$

We have to show that $(a_1, b_1) \leq_{\text{lex}} (a_3, b_3)$. The proof proceeds by case distinction.

(1) **and** (3)**:** We have $a_1 \prec a_2$ and $a_2 \prec a_3$. Since $\preceq$ is transitive, it follows that $a_1 \prec a_3$. Hence, $(a_1, b_1) \leq_{\text{lex}} (a_3, b_3)$.

(1) **and** (4)**:** We have $a_1 \prec a_2$ and $a_2 = a_3 \wedge b_2 \sqsubseteq b_3$. Hence, $a_1 \prec a_3$ and, therefore, $(a_1, b_1) \leq_{\text{lex}} (a_3, b_3)$.

(2) **and** (3)**:** We have $a_1 = a_2 \wedge b_1 \sqsubseteq b_2$ and $a_2 \prec a_3$. Hence, $a_1 \prec a_3$ and, therefore, $(a_1, b_1) \leq_{\text{lex}} (a_3, b_3)$.

(2) **and** (4)**:** We have $a_1 = a_2 \wedge b_1 \sqsubseteq b_2$ and $a_2 = a_3 \wedge b_2 \sqsubseteq b_3$. It follows that $a_1 = a_3$. Since $\sqsubseteq$ is transitive, we also have $b_1 \sqsubseteq b_3$. Therefore, $(a_1, b_1) \leq_{\text{lex}} (a_3, b_3)$.

*(3 Points)*

### 6.4 Countability

**a)** **i)** The set of all Java programs is countable. Every Java program can be seen as a finite binary sequence. That is, there is an injection from the set of all Java programs to the set $\{0,1\}^*$ of finite binary sequences. By Theorem 3.15, this set is countable.

**ii)** This set is uncountable. The proof is very similar to the proof of Theorem 3.20. Assume that there is a bijection $f : \mathbb{N} \to A$. Let $\beta_{i,j}$ denote the $j$-th number in the $i$-th sequence. We define a new sequence as follows:

$$\alpha := R_{10}(\beta_{0,0}+1), R_{10}(\beta_{1,1}+1), R_{10}(\beta_{3,3}+1), \ldots$$

where $R_{10}(a)$ denotes the remainder when $a$ is divided by 10. Of course, $\alpha \in A$. Moreover, there is no $n \in \mathbb{N}$ such that $\alpha = f(n)$, since $\alpha$ disagrees with a sequence $f(n)$ on position $n$.

**iii)** This set is uncountable. We can define an injective function $f : [0,1] \to C$ by $f(x) = \left(x, \sqrt{1-x^2}\right)$. Hence, we have $[0,1] \preceq C$. The fact that the interval $[0,1]$ is uncountable follows from Theorem 3.20 and the fact that any element of $\{0,1\}^\infty$ can be interpreted as the binary expansion of a number in the interval $[0,1]$, and vice versa.

**iv)** This set is uncountable. To show that this must be the case, it is enough to consider the number of possible equivalence classes of 0. Every equivalence relation on $\mathbb{N}$ defines the equivalence class of 0, so the number of such relations must be greater than the number of different equivalence classes of 0. If we can show that the set of possible equivalence classes of 0 is uncountable, the claim follows. Notice now that each equivalence class of 0 is simply a subset of $\mathbb{N}$. Since the number of subsets of $\mathbb{N}$ is uncountable (this fact follows from Theorem 3.20 and the fact that each subset of $\mathbb{N}$ corresponds to a binary sequence in $\{0,1\}^\infty$, where 1 at position $i$ means that $i$ is in a given subset, while 0 means that it is not), the set of all equivalence classes of 0 is uncountable as well.

**b)** At any point in time $t$ we can fire a torpedo to position $s = x \cdot t + y$ for some $x$ and $y$. The submarine sinks if its speed and starting position happened to be $x$ and $y$. Thus, at any time $t$ we can make a guess about $x$ and $y$ and sink the submarine based on that guess. We now have to systematically check all the pairs $(x,y) \in \mathbb{Z} \times \mathbb{Z}$.

Hence, we need a surjective function $f : \mathbb{N} \to \mathbb{Z} \times \mathbb{Z}$ that will assign to a time $t$ a pair $(x,y)$. (Surjectivity guarantees that every $(x,y)$ will be tested at some time $t'$.) Since $\mathbb{Z}\times\mathbb{Z}$ is countable (by Example 3.66 and Corollary 3.17), there exists an injective function $g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{N}$. We can now define $f$ as

$$f(n) := \begin{cases} (a,b) & \textbf{if} \quad \exists (a,b) \;\; g((a,b)) = n \\ (0,0) & \textbf{otherwise} \end{cases}$$

By the injectivity of $g$, we have $\{(a,b)\} = g^{-1}(\{g((a,b))\})$ for all $(a,b) \in \mathbb{Z} \times \mathbb{Z}$. Also, for any $(a,b)$ there exists an $n \in \mathbb{N}$ such that $g((a,b)) = n$ and, therefore, there exists an $n \in \mathbb{N}$ such that $f(n) = (a,b)$. Hence, $f$ is surjective and we will eventually sink the submarine.

### 6.5 A property of sets

We define inductively an infinite sequence of sets $X_0, X_1, \ldots$ such that $X_n \subseteq \mathcal{P}(X_n)$ for all $n \in \mathbb{N}$. Namely, we set $X_0 = \varnothing$ and $X_{n+1} := X_n \cup \{X_n\}$ for $n > 0$.

In the following, we prove by induction that this sequence fulfills the requirement. Additionally, we prove that for all $n \geq 0$, we have $|X_n| = n$ and that for all $y \in X_n$, $|y| < |X_n|$. From the former property it follows that the sets $X_0, X_1, \ldots$ are distinct. The latter property is a technical fact for the proof.

**Basis** Obviously, we have $\varnothing \subseteq \mathcal{P}(\varnothing)$ and $|\varnothing| = 0$. The last property holds, because there is no $y \in \varnothing$.

**Induction step** Fix any $n \geq 0$ and assume that (a) $X_n \subseteq \mathcal{P}(X_n)$, (b) $|X_n| = n$ and (c) for all sets $y \in X_n$, $|y| < |X_n|$.

The statement (b) must hold for $X_{n+1} = X_n \cup \{X_n\}$, because $|X_n| = n$ by the property (b) of $X_n$ and $X_n \notin X_n$ by the property (c) of $X_n$.

The property (c) of $X_{n+1}$ follows from the fact that for any $y \in X_{n+1}$, we have $|y| \leq |X_n|$ (by the property (c) of $X_n$) and $|X_n| < |X_{n+1}|$ (by the property (b) of $X_n$ and $X_{n+1}$).

To prove that $X_{n+1}$ has the property (a), notice that $\mathcal{P}(X_n) \subseteq \mathcal{P}(X_{n+1})$, since $X_n \subseteq X_{n+1}$. Hence, by the property (a) of $X_n$, we have $X_n \subseteq \mathcal{P}(X_n) \subseteq \mathcal{P}(X_{n+1})$. By the definition of the power set, $\{X_n\} \subseteq \mathcal{P}(X_{n+1})$. Therefore, $X_n \cup \{X_n\} \subseteq \mathcal{P}(X_{n+1})$.