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Diskrete Mathematik

Solution 5

5.1 Family relations

a) The relations can be expressed in the following way:

i) $iggf = if \circ ip \circ ip$

ii) $ihs = (ic \circ ip) \setminus (ic \circ im \cap ic \circ if)$

iii) $ico = (ic \circ ic \circ ip \circ ip) \setminus (ic \circ ip)$

b) These relations are neither the same, nor is one a subset of the other. To see this, consider six different people a, b, c, d, e and f, where c and d are the mother and father of a, while e and f are the mother and father of b. Also, c has no common parent or child with e or f and f has no common parent or child with c or d.

For such six people, $a \; ic \circ ic \circ ip \circ ip \; b$ if and only if d and e have a common parent, while $a \; ic \circ ip \circ ic \circ ip \; b$ if and only if d and e have a common child.

While it is theoretically possible that two people share both a parent and a child, in general neither of these implies the other. One can easily argue that a counterexample actually exists.

5.2 Operations on relations

i) Two numbers (a, c) are in the relation whenever there exists a b such that $a < b$ and $b \mid c$. This relation is not reflexive, since $(1, 1) \notin < \circ \mid$. Moreover, it is not symmetric, because $(1, 2) \in < \circ \mid$, but $(2, 1) \notin < \circ \mid$. Finally, it is also not transitive, since $(2, 0) \in < \circ \mid$ and $(0, 1) \in < \circ \mid$, but $(2, 1) \notin < \circ \mid$.

ii) Two numbers (a, b) are in the relation whenever $a \mid b$ or $a \equiv_2 b$. This relation is reflexive, since for any a, we have $a \equiv_2 a$ (alternatively, one could use the fact that $a \mid a$). It is, however, not symmetric, because $(1, 2) \in \; \mid \cup \equiv_2$, but $(2, 1) \notin \; \mid \cup \equiv_2$. It is also not transitive, since $(3, 1) \in \; \mid \cup \equiv_2$ and $(1, 2) \in \; \mid \cup \equiv_2$, but $(3, 2) \notin \; \mid \cup \equiv_2$.

iii) Two numbers (a, b) are in the relation whenever $a \mid b$ or $b \mid a$. This relation is reflexive, since for any a, we have $a \mid a$. It is also symmetric, because for any (a, b), we trivially have $a \mid b$ or $b \mid a$ if and only if $b \mid a$ or $a \mid b$. The relation is, however, not transitive, since $(3, 1) \in \mid \cup \mid^{-1}$ and $(1, 2) \in \mid \cup \mid^{-1}$ but $(3, 2) \notin \mid \cup \mid^{-1}$.

	Relation	reflexive	symmetric	transitive
i)	$< \circ \mid$	✗	✗	✗
ii)	$\mid \cup \equiv_2$	✓	✗	✗
iii)	$\mid \cup \mid^{-1}$	✓	✓	✗

5.3 Properties of relations

a) We have $\rho^3 = \{(1,1),(1,3),(2,2),(4,4)\}$ *(1 Point)*

and

$$M^{\rho^*} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

(1 Point)

b) The statement is false. A counterexample is the relation $\sigma = \{(0,1),(1,0)\}$ on the set $A = \{0,1\}$. *(1 Point)*

Obviously, σ is not reflexive. Further, we have $\sigma^2 = \{(0,0),(1,1)\}$, which is reflexive. This disproves the statement. *(1 Point)*

c) The statement is true. We present two different (direct) proofs, each using one of the two equivalent definitions of an antisymmetric relation.

Proof 1

Let A be any set and let σ and ρ be any antisymmetric relations on A. We show that $\sigma \cap \rho$ is antisymmetric, that is that for any $(a,b) \in A \times A$, we have

$$a\ (\sigma \cap \rho)\ b \ \wedge\ b\ (\sigma \cap \rho)\ a \ \Rightarrow a = b.$$

(1 Point)

To this end, consider any pair $(a,b) \in A \times A$ such that $a\ (\sigma \cap \rho)\ b$ and $b\ (\sigma \cap \rho)\ a$. From $a\ (\sigma \cap \rho)\ b$ it follows that $a\ \sigma\ b$. Moreover, from $b\ (\sigma \cap \rho)\ a$ it follows that $b\ \sigma\ a$. *(1 Point)*

From those two facts and the antisymmetry of σ we can conclude that $a = b$. *(1 Point)*

Proof 2 Let A be any set and let σ and ρ be any antisymmetric relations on A. We show that $\sigma \cap \rho$ is antisymmetric, that is that $(\sigma \cap \rho) \cap (\sigma \cap \rho)^{-1} \subseteq \mathsf{id}$. *(1 Point)*

$$\begin{aligned}
(\sigma \cap \rho) \cap (\sigma \cap \rho)^{-1} &= \sigma \cap \rho \cap (\sigma \cap \rho)^{-1} && \\
&\subseteq \rho \cap (\sigma \cap \rho)^{-1} && (A \cap B \subseteq B) \\
&\subseteq \rho \cap \rho^{-1} && (\alpha \subseteq \beta \Rightarrow \alpha^{-1} \subseteq \beta^{-1}) \\
&\subseteq \mathsf{id} && (\text{antisymmetry of } \sigma)
\end{aligned}$$

Therefore, $\sigma \cap \rho$ is antisymmetric. *(1 Point)*

The statement $\alpha \subseteq \beta \Rightarrow \alpha^{-1} \subseteq \beta^{-1}$ follows directly from the definition of the inverse of a relation. *(1 Point)*

5.4 A false proof

a) For an arbitrary $x \in A$, there does not always exist a $y \in A$ such that $x\ \rho\ y$.

b) Consider the following counterexample: $A = \{1,2\}$ and $\rho = \{(1,1)\}$. The relation ρ is symmetric and transitive. However, it is not reflexive, since $2\ \rho\ 2$ does not hold.

5.5 Equivalence Relations

a) To prove that $\sim$ is an equivalence relation, we have to show that it has the following properties:

Reflexivity: For any point $(a, b) \in \mathbb{R}$, we have

$$\big((a-x)^2+b^2\big)\big((a+x)^2+b^2\big)-\big((a-x)^2+b^2\big)\big((a+x)^2+b^2\big)=0.$$

This means that $(a, b) \sim (a, b)$. Therefore, $\sim$ is reflexive.

Symmetry: For any two points $(a, b), (c, d) \in \mathbb{R}^2$ such that $(a, b) \sim (c, d)$, that is

$$\big((a-x)^2+b^2\big)\big((a+x)^2+b^2\big)-\big((c-x)^2+d^2\big)\big((c+x)^2+d^2\big)=0.$$

we have

$$\big((c-x)^2+d^2\big)\big((c+x)^2+d^2\big)-\big((a-x)^2+b^2\big)\big((a+x)^2+b^2\big)=0$$

This means that $(c, d) \sim (a, b)$. Therefore, $\sim$ is symmetric.

Transitivity: Take any three points $(a, b), (c, d), (e, f) \in \mathbb{R}^2$ such that $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. This means that

$$\big((a-x)^2+b^2\big)\big((a+x)^2+b^2\big)-\big((c-x)^2+d^2\big)\big((c+x)^2+d^2\big)=0$$

and

$$\big((c-x)^2+d^2\big)\big((c+x)^2+d^2\big)-\big((e-x)^2+f^2\big)\big((e+x)^2+f^2\big)=0$$

It follows that

$$\begin{aligned}
&\big((a-x)^2+b^2\big)\big((a+x)^2+b^2\big)-\big((e-x)^2+f^2\big)\big((e+x)^2+f^2\big)\\
=&\big((a-x)^2+b^2\big)\big((a+x)^2+b^2\big)-\big((c-x)^2+d^2\big)\big((c+x)^2+d^2\big)\\
&+\big((c-x)^2+d^2\big)\big((c+x)^2+d^2\big)-\big((e-x)^2+f^2\big)\big((e+x)^2+f^2\big)\\
=&\,0+0=0
\end{aligned}$$

This means that $(a, b) \sim (e, f)$. Therefore, $\sim$ is transitive.

b) First, notice that $(a-x)^2+b^2$ is the squared distance from the point (a, b) to $(x, 0)$. Analogously, $(a+x)^2+b^2$ is the squared distance from (a, b) to $(-x, 0)$. Therefore, an equivalence class of $\sim$ contains all points P with the same product of the distance from P to $(-x, 0)$ and the distance from P to $(x, 0)$.

Such equivalence classes are also known as the Cassini ovals. A special case of $[(0, 0)]$ for $x \neq 0$ is called a lemniscate.